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MONOTONE DECOMPOSITIONS OF IUC CONTINUA

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ABSTRACT. For the class of hereditarily unicoherent metric continua a spectrum of monotone decompositions has been developed by several authors which "improves" the quotient spaces. This spectrum is developed for a broader class of continua, namely continua with property IUC. A metric continuum M has property IUC provided each proper subcontinuum of M with interior is unicoherent. One important result which develops is that semiaposyndetic IUC continua are hereditarily arcwise connected. Also the notion of smoothness is studied for IUC continua.

0. Introduction. In [**5**] FitzGerald and Swingle, by use of set functions, described a monotone upper semicontinuous decomposition $\mathcal{D}_a$ of a compact Hausdorff continuum M such that $\mathcal{D}_a$ is the core decomposition of M with respect to having an aposyndetic quotient space. A spectrum of decompositions which "fills in" between M and $M/\mathcal{D}_a$ has been developed for hereditarily unicoherent metric continua by such authors as J. J. Charatonik [**1**], E. J. Vought [**8, 12**] and G. R. Gordh [**6, 8**].

The main purpose of this paper is to extend the spectrum of decompositions to the class of IUC continua. A compact metric continuum has *property IUC* provided each proper subcontinuum with interior is unicoherent [**3**]. If the continuum has property IUC hereditarily then the continuum has *property HIUC*. Note that the class of IUC continua includes all atriodic continua and hereditarily unicoherent continua.

Throughout this paper M will denote a compact metric continuum. An upper semicontinuous (u.s.c.) decomposition $\mathcal{D}$ of M is *core* with respect to some property P provided $\mathcal{D}$ is the unique minimal decomposition with respect to which $\mathcal{D}$ has property P. A set function N is *expansive* provided that if A and B are subsets of M then $A \subseteq N(A)$ and $N(A) \subseteq N(B)$ whenever $A \subseteq B$. The subset A is N-closed provided $A = N(A)$. In [**5**, Theorem 2.5, p. 37] FitzGerald and Swingle prove that if N is any expansive set function on M then there exists a core decomposition $\mathcal{G}$ of M with respect to the property: $\mathcal{G}$ is u.s.c with N-closed elements. They also note that if N is monotone then $\mathcal{G}$ is monotone.

Let $\mathcal{D}$ be a u.s.c. decomposition of M and $\mathcal{K} \subseteq \mathcal{D}$. We denote by $\mathcal{K}^*$ the subset of M consisting of the sum of the elements of $\mathcal{K}$. If B is a subset of $M/\mathcal{D}$ then B^{-1} will

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denote $f^{-1}(B)$, where f is the quotient map from M onto $M/\mathfrak{D}$. If $\mathcal{K} \subseteq \mathfrak{D}$ and $X \subseteq M$ then $\mathcal{K} \cap X$ will denote $\{k \cap X \mid k \in \mathcal{K}\}$, the restriction of $\mathcal{K}$ to X.

Finally, it should be noted that property IUC is preserved by monotone mappings, as this fact will be used throughout this paper.

I. Decompositions and hereditary decomposability. For $A \subseteq M$ let $V(A)$ denote the closure of the sum of A and the indecomposable subcontinua of M which intersect A. Now V is expansive and montone. Let $\mathfrak{D}_V$ denote the monotone u.s.c. core decomposition of M into V-closed elements. Note that $M/\mathfrak{D}_V$ is hereditarily decomposable.

THEOREM 1.1. *Suppose that M has property HIUC and $\mathfrak{D}$ is a monotone u.s.c decomposition of M. If $M/\mathfrak{D}$ is hereditarily decomposable then each element of $\mathfrak{D}$ is V-closed.*

PROOF. Let I be an indecomposable subcontinuum of M and $\mathcal{K} = \{d \in \mathfrak{D} \mid d$ intersects $I\}$. If $\mathcal{K}$ is degenerate then I is contained in the element of $\mathcal{K}$. So suppose that $\mathcal{K}$ is nondegenerate. Now $\mathcal{K} \cap I$ is a u.s.c. decomposition of I and since $I/(\mathcal{K} \cap I) = \mathcal{K}^*/\mathcal{K}$ which is hereditarily decomposable, some element d_1 of $\mathcal{K}$ has a nonconnected intersection with I.

Let $d \in \mathcal{K} - \{d_1\}$ and suppose that $d^{-1} \not\subseteq I$. Then $d_1^{-1} \cup I$ is a nonunicoherent proper subcontinuum of $d_1^{-1} \cup I \cup d^{-1}$ with interior respect to $d_1^{-1} \cup I \cup d^{-1}$, a contradiction. Hence $d^{-1} \subseteq I$ and each element of $\mathcal{K} - \{d_1\}$ must lie in a composant of I. Then by [**4**] either (1) d_1^{-1} hits every composant of I or (2) d_1^{-1} misses uncountably many composants.

Suppose (1). By [**4**, Theorem 8, p. 40] some composant C is hit by d_1^{-1} in a nonconnected intersection. Let A and B be components of $C \cap d_1^{-1}$ and let N be a subcontinuum of C containing A and B. Then $N \cup d_1^{-1}$ is a nonunicoherent proper subcontinuum of $I \cup d_1^{-1}$ with interior with respect to $I \cup d_1^{-1}$.

Suppose (2). Let C be a composant of I which misses d_1^{-1}. We show that $C/(\mathcal{K} \cap C)$ is a composant of $I/(\mathcal{K} \cap I)$. Let x and z be points of $C/(\mathcal{K} \cap C)$ and N be a subcontinuum of C containing x^{-1} and z^{-1}. Hence $f(N)$ is a subcontinuum of $C/(\mathcal{K} \cap C)$ containing x and z, where f is the induced quotient map from M to $M/\mathfrak{D}$. Let $y \in I/(\mathcal{K} \cap I) - C/(\mathcal{K} \cap C)$ and S be a subcontinuum of $I/(\mathcal{K} \cap I)$ from x to y. Let T be a component of $S - \{d_1\}$. Hence $\mathrm{cl}(T^{-1})$, the closure of T^{-1}, is a subcontinuum of I intersecting two composants and therefore $\mathrm{cl}(T^{-1}) = I$. Hence $S = I/(\mathcal{K} \cap I)$ and $C/(\mathcal{K} \cap C)$ is a composant of $I/(\mathcal{K} \cap I)$.

Since there are uncountably many composants of I missing d_1^{-1} there are uncountably many composants of $I/(\mathcal{K} \cap I)$, a contradiction. Hence the theorem is established.

COROLLARY 1.2. *If M has property HIUC then M admits a monotone core decomposition, $\mathfrak{D}_{hd}$, with respect to the quotient space being hereditarily decomposable, and this decomposition is the same as $\mathfrak{D}_V$.*

One cannot weaken the hypothesis of Theorem 1.1 to IUC continua as will be seen in the following

EXAMPLE 1.3. Let K denote the sum of the Knaster indecomposable continuum and an arc A which intersects each composant. Let M be the sum of K and a half-ray which limits on K. Hence M has property IUC but not HIUC. Let $\mathfrak{D}$ be the monotone u.s.c. decomposition of M induced by collapsing the arc A to a single point. Hence $M/\mathfrak{D}$ is hereditarily decomposable, but K fails to lie in a single element of $\mathfrak{D}$.

II. Decompositions and arcwise connectivity. If I is an irreducible continuum there exists a monotone core decomposition $\mathcal{G}$ of I such that $I/\mathcal{G}$ is an arc or degenerate. The elements of the decomposition $\mathcal{G}$ are called the *layers* of I.

DEFINITION 2.1 (CHARATONIK). A monotone u.s.c. decomposition $\mathfrak{D}$ of M is admissible provided each layer of every irreducible subcontinuum of M lies in an element of $\mathfrak{D}$.

If $A \subseteq M$ then $C(A)$ denotes the closure of the sum of all layers of irreducible subcontinua which intersect A. Clearly C is expansive and monotone. Let $\mathfrak{D}_C$ denote the core decomposition of M into C-closed elements. Hence $M/\mathfrak{D}_C$ is hereditarily arcwise connected since each layer of every irreducible subcontinuum lies in an element of $\mathfrak{D}_C$. Note that any decomposition of M into C-closed elements is admissible.

THEOREM 2.2. *If M has property HIUC and $\mathfrak{D}$ is a monotone u.s.c. decomposition of M such that $M/\mathfrak{D}$ is hereditarily arcwise connected, then each element of $\mathfrak{D}$ is C-closed.*

PROOF. Let I be an irreducible subcontinuum of M.

Case 1. I is its own layer. In [**11**] E. J. Vought described the structure of the layers of irreducible continua that have nonvoid interior. An outline of his technique follows.

Let I be an irreducible continuum and J be a layer of I with interior. Let $I_1, I_2, \ldots$ denote the indecomposable subcontinua of J with interior. Let $\mathrm{Ch}_0(I_i) = I_i$ for each positive integer i. For each nonlimit ordinal α define

$$\mathrm{Ch}_\alpha(I_i) = \mathrm{cl}\{y \in J \mid y \text{ can be } \mathrm{Ch}_{\alpha-1}\text{-chained to } I_i\}$$

(if F is a set function then y can be F-chained to $R \subseteq J$ if there exists $I_{i_1}, I_{i_2}, \ldots, I_{i_n}$ such that $F(I_{i_1}), F(I_{i_2}), \ldots, F(I_{i_n})$ is a simple chain where $y \in F(I_{i_1})$ and R intersects $F(I_{i_n})$). If α is a limit ordinal, define $\mathrm{Ch}_\alpha(I_i) = \mathrm{cl}(\bigcup_{\beta<\alpha} \mathrm{Ch}_\beta(I_i))$. Let γ be a countable ordinal such that $\mathrm{Ch}_\gamma(I_i) = \mathrm{Ch}_{\gamma+1}(I_i)$ for all i. He then proves that $\mathrm{cl}(J^0) = \mathrm{Ch}_\gamma(I_1)$.

Hence we assume that $I = J$ and hence $I = \mathrm{Ch}_\gamma(I_1)$. Since $M/\mathfrak{D}$ is hereditarily arcwise connected it is hereditarily decomposable and, by Theorem 1.1, each indecomposable subcontinuum is contained in an element of $\mathfrak{D}$ and for each ordinal $\beta \leqslant \gamma$ so is $\mathrm{Ch}_\beta(I_1)$.

Case 2. The collection of layers $\mathfrak{L}$ of I is nondegenerate. Let $\mathcal{K} = \{d \in \mathfrak{D} \mid d$ intersects $I\}$. Hence $\mathcal{K}^*/\mathcal{K} = I/(\mathcal{K} \cap I)$ is a hereditarily arcwise connected subcontinuum of $M/\mathfrak{D}$. If $\mathcal{K} \cap I$ is monotone, then $I/(\mathcal{K} \cap I)$ is irreducible and is hence either degenerate or an arc. But by [**1**, Corollary 8, p. 123] $\mathcal{K} \cap I$ is admissible, and hence $\mathfrak{D}$ is also.

So we suppose that $d_1 \in \mathcal{K}$ such that $d_1^{-1} \cap I$ is not connected. As in the proof of Theorem 1.1 we assume that $d^{-1} \subseteq I$ for each $d \in \mathcal{K} - \{d_1\}$.

Let $\mathfrak{L} = \{L_x\}$ where L_x denotes the layer of I containing the point x of I. If I is irreducible from the point a to the point b then $\mathfrak{L}^*/\mathfrak{L}$ is an arc from L_a to L_b. Denote by $[L_a, L_b]$ the arc $\mathfrak{L}^*/\mathfrak{L}$ in $M/\mathfrak{D}$.

Note that (1) d_1^{-1} intersects L_a^{-1} and L_b^{-1} and $d_1^{-1} \cap L_x^{-1}$ is a continuum for each x in I. Since I is irreducible and $d_1^{-1} \cap I$ is not connected, no component of $d_1^{-1} \cap I$ meets both L_a^{-1} and L_b^{-1}.

Also $d_1^{-1} \cap I$ is the sum of two disjoint continua A and B where A intersects L_a^{-1} and B intersects L_b^{-1}. For if A and B are components of $d_1^{-1} \cap I$ intersecting L_a^{-1} and L_b^{-1}, respectively, then by (1) these are the only two such components. Suppose C is a component of $d_1^{-1} \cap I$ different from A and B. Then C misses $L_a^{-1} \cup L_b^{-1}$. Let L_z be in $\mathfrak{L}$ such that C intersects $[L_a, L_z]^{-1} \subseteq [L_a, L_b)^{-1}$. Hence $d_1^{-1} \cup [L_a, L_z]^{-1}$ is a nonunicoherent proper subcontinuum of $d_1^{-1} \cup I$ with interior with respect to $d_1^{-1} \cup I$. Likewise we may suppose that $A \subseteq L_a^{-1}$ and $B \subseteq L_b^{-1}$.

Let $\mathcal{K}' = \{C \mid C$ is a component of $\mathcal{K} \cap I\}$. We show that $I/\mathcal{K}'$ is arcwise connected. Let p be a point of $(L_a, L_b)^{-1}$. Hence $[L_a, L_p]^{-1}$ and $[L_p, L_b]^{-1}$ are subcontinua of I containing p and, respectively, A and B. Let A_p and B_p be arcs in $I/(\mathcal{K} \cap I)$ from the element containing p to $d_1 \cap I$ contained in $f([L_a, L_p]^{-1})$ and $f([L_p, L_b]^{-1})$, respectively, where f is the induced quotient map for $\mathcal{K} \cap I$.

Now A_p^{-1} cannot meet both L_a^{-1} and L_b^{-1} since I is irreducible, and hence A_p is an arc in $I/\mathcal{K}'$ from the element containing p to A. Likewise B_p is an arc in $I/\mathcal{K}'$ from L_p to B. If p is in $L_a^{-1} - A^{-1}$ then let A_p be an arc in $I/(\mathcal{K} \cap I)$ from the element containing p to A contained in $f(L_a)$. Hence A_p is in $I/\mathcal{K}'$ and $I/\mathcal{K}'$ is arcwise connected. But since $I/\mathcal{K}'$ is irreducible, it is an arc. Therefore by [**1**], $\mathcal{K}'$ is admissible and hence so is $\mathcal{K} \cap I$. Therefore $\mathfrak{D}$ is admissible.

Corollary 2.3. *The HIUC continuum M admits a core decomposition $\mathfrak{D}_{hac}$ with respect to the quotient space being hereditarily arcwise connected, and this decomposition is exactly $\mathfrak{D}_C$.*

Corollary 2.4. *$\mathfrak{D}_V \leqslant \mathfrak{D}_C$ and if M has property HIUC then $\mathfrak{D}_{hd} \leqslant \mathfrak{D}_{hac}$, where "$\leqslant$" means "refines".*

Question 2.5. Is Theorem 2.2 true for continua with property IUC?

III. Decompositions and semiaposyndesis. The continuum M is *aposyndetic* at the subset $A \subseteq M$ with respect to $B \subseteq M$ provided there exists a subcontinuum H of M containing A in its interior that misses B. The continuum M is aposyndetic provided M is aposyndetic at each point with respect to every other point of M. Also, M is *semiaposyndetic* provided for every two points of M, the continuum M is aposyndetic at one of the points with respect to the other. Let $T(A) = A \cup \{p \in M \mid M$ is not aposyndetic at p with respect to $A\}$ and $K(A) = A \cup \{p \in M \mid M$ is not aposyndetic at A with respect to $p\}$. It is known that $T(A)$ is a continuum provided A is connected, but this is not true for $K(A)$ in general.

Lemma 3.1. *If X is an indecomposable subcontinuum of the IUC continuum M then $X \subseteq K(x)$ for each x in X.*

PROOF. Let $x \in X$ and suppose there exists a point w of X and a subcontinuum H of M such that $x \in H^0 \subseteq H \subseteq M - \{w\}$. Hence there exists a composant C of X and a component K of $C - H$ such that $\mathrm{cl}(K) \cap H$ is not connected. But then $\mathrm{cl}(K) \cup H$ is a nonunicoherent proper subcontinuum of M with interior, a contradiction. Hence $X \subseteq K(x)$.

THEOREM 3.2. *If M has property IUC and $x \in M$ then $K(x)$ is a subcontinuum of M.*

PROOF. Clearly $K(x)$ is closed. Suppose that $K(x)$ is the sum of two mutually exclusive point sets A and B with $x \in A$. Let $y \in B$ and I be an irreducible subcontinuum of M from x to y. Hence I contains a point z not in $K(x)$. Let H_z be a subcontinuum of M such that $x \in H_z^0 \subseteq H_z \subseteq M - \{z\}$. Hence $y \in H_z$ and $I \cap H_z$ is not connected. Since M has property IUC we have that $M = I \cup H_z$.

Therefore by [3, Theorem 3.1] M is nonunicoherent and by [3, Corollary 3.2] $M - I$ is connected. We show that $\mathrm{cl}(M - I)$ is indecomposable. Suppose rather that $\mathrm{cl}(M - I)$ is the sum of the proper subcontinua C and D. Since M has property IUC the set $C \cap D$ is a subcontinuum of M which misses $I \cap H_z$. Let $\mathcal{C}_z = \{k \mid k$ is a component of $H_z - \mathrm{cl}(M - I)$ whose closure intersects $C\}$ and $\mathcal{D}_z = \{k \mid k$ is a component of $H_z - \mathrm{cl}(M - I)$ whose closure intersects $D\}$. Since M has property IUC the sets $\mathcal{C}_z^*$ and $\mathcal{D}_z^*$ are mutually separated. Each of $C \cup \mathcal{C}_z^*$ and $D \cup \mathcal{D}_z^*$ is a subcontinuum of M, and one must contain x in its interior and miss y. Hence $\mathrm{cl}(M - I)$ is indecomposable, and by Lemma 3.1 $\mathrm{cl}(M - I) \subseteq K(w)$ for each $w \in \mathrm{cl}(M - I)$.

If $x \in \mathrm{cl}(M - I)$ then $y \notin \mathrm{cl}(M - I)$ and each of x and y is in

$$\mathrm{cl}(M - \mathrm{cl}(M - I)).$$

Since I is irreducible, and by [3, Corollary 3.2] $M - \mathrm{cl}(M - I)$ is connected since M is not unicoherent, we have that $I = \mathrm{cl}(M - \mathrm{cl}(M - I))$. Suppose that I is the sum of the two proper subcontinua E and F. Then $E \cap F$ misses $I \cap \mathrm{cl}(M - I)$ and we may assume that $x \in E - (E \cap F)$ and $y \in F - (E \cap F)$. Then $E \cup \mathrm{cl}(M - I)$ contains x in its interior and misses y. Hence I is indecomposable and $I \subseteq K(x)$, a contradiction.

If $x \notin \mathrm{cl}(M - I)$, by the above procedure we have that $\mathrm{cl}(M - \mathrm{cl}(M - I))$ is indecomposable and by Lemma 3.1 is in $K(x)$. But then $M = K(x)$, a contradiction. Hence $K(x)$ is a continuum.

COROLLARY 3.3. *If A is a subcontinuum of the IUC continuum M then so is $K(A)$.*

DEFINITION 3.4. Let $Z(A) = T(A) \cap K(A)$. Hence Z is an expansive set function.

LEMMA 3.5. *Let x be a point of the IUC continuum M. Then $Z(x)$ is a continuum.*

PROOF. Clearly $Z(x)$ is closed. Suppose that $Z(x)$ is the sum of two mutually exclusive closed sets A and B. Since M has property IUC the continuum $T(x) \cup K(x)$ is M or has void interior.

Case 1. $M = T(x) \cup K(x)$. Hence M is not unicoherent and by [3] $M - K(x)$ is connected. Let y be a point of $M - K(x)$. Hence $y \in T(x)$ and $x \in \mathrm{cl}(M - K(x))$.

As in the proof of Theorem 3.2 $\mathrm{cl}(M - K(x))$ is indecomposable and hence $\mathrm{cl}(M - K(x)) \subseteq K(x)$, a contradiction.

Case 2. $T(x) \cup K(x)$ has void interior. We may suppose that $x \in A$ and let y be a point of B. Let I be a subcontinuum of $T(x)$ irreducible from x to y, and w be a point of $I - (I \cap K(x))$. Hence there exists a subcontinuum H_w of M such that $x \in H_w^0 \subseteq H_w \subseteq M - \{w\}$. Now $I \cap H_w$ cannot be connected since $y \in H_w$ and I is irreducible from x to y. Hence $H_w \cup I$ is a nonunicoherent proper subcontinuum of M with interior, a contradiction. Hence $Z(x)$ is a continuum.

COROLLARY 3.6. *If A is a subcontinuum of the IUC continuum M then so is $Z(A)$.*

Let $\mathfrak{D}_Z$ denote the monotone core decomposition of M with respect to having Z-closed elements.

THEOREM 3.7. *If M has property IUC then $M/\mathfrak{D}_Z$ is semiaposyndetic.*

PROOF. Let d_x and d_y be points of $M/\mathfrak{D}_Z$. We will consider two cases.

Case 1. $T(d_x^{-1})$ does not intersect d_y^{-1}. If $b \in d_y^{-1}$ there exists a continuum H_b such that $b \in H_b^0 \subseteq H_b \subseteq M - d_x^{-1}$. Since d_y^{-1} is compact there exists a continuum H such that $d_y^{-1} \subseteq H^0 \subseteq H \subseteq M - d_x^{-1}$. Hence by the upper semicontinuity of $\mathfrak{D}_Z$, $f(H)$ is a subcontinuum of $M/\mathfrak{D}_Z$ containing d_y in its interior and missing d_x (where f is the quotient map from M to $M/\mathfrak{D}_Z$).

Case 2. $T(d_x^{-1})$ intersects d_y^{-1}. We may suppose that $T(d_y^{-1})$ intersects d_x^{-1}. Let I be a subcontinuum of $T(d_x^{-1})$ irreducible between d_x^{-1} and d_y^{-1}, and let $b \in I \cap d_y^{-1}$. Hence $b \notin Z(d_x^{-1})$ and $b \notin K(d_x^{-1})$. There exists a subcontinuum H_b of M such that $d_x^{-1} \subseteq H_b^0 \subseteq H_b \subseteq M - \{b\}$. Now H_b intersects d_y^{-1} but misses points of $I - (I \cap d_y^{-1})$ and hence $(H_b \cup d_y^{-1}) \cap I$ is not connected.

Since M has property IUC we have $M = H_b \cup d_y^{-1} \cup I$ and by [3] M is not unicoherent, and $Q = M - (d_x^{-1} \cup I \cup d_y^{-1})$ is connected. Now $\mathrm{bdy}(\mathrm{cl}(Q))$ is not connected and $\mathrm{cl}(Q)$ is irreducible about its boundary. If each of I and $\mathrm{cl}(Q)$ is decomposable, then $I = I_x \cup I_y$ and $\mathrm{cl}(Q) = Q_x \cup Q_y$ where $I_x \cap I_y$ misses $d_x^{-1} \cup d_y^{-1}$ and $Q_x \cap Q_y$ misses $\mathrm{bdy}(\mathrm{cl}(Q))$. We may assume that each of I_y and Q_y misses d_x^{-1} since M has property IUC. Therefore $d_y^{-1} \cup I_y \cup Q_y$ contains b in its interior and misses d_x^{-1}, contradicting the fact that $b \in T(d_x^{-1})$.

Suppose that I is indecomposable. As in the proof of Theorem 3.2 $b \in I \subseteq K(d_x^{-1})$, a contradiction. Suppose I is decomposable. Then $b \in Q$, for if not, $I_y \cup d_y^{-1}$ contradicts the fact that $b \in T(d_x^{-1})$. Again $b \in Q \subseteq K(d_x^{-1})$.

THEOREM 3.8. *If $\mathfrak{D}$ is a monotone u.s.c. decomposition of the IUC continuum M such that $M/\mathfrak{D}$ is semiaposyndetic then the elements of $\mathfrak{D}$ are Z-closed.*

PROOF. Let $d \in \mathfrak{D}$ and since Z is expansive $d^{-1} \subseteq Z(d^{-1})$. Suppose there exists a point $w \in Z(d^{-1}) - d^{-1}$. Hence the element of $\mathfrak{D}$ containing w, d_w, is different from d. Since $M/\mathfrak{D}$ is semiaposyndetic there exists a subcontinuum of $M/\mathfrak{D}$ containing one of d and d_w in its interior and missing the other point. Since $w \in K(d^{-1})$ we may suppose that there is a subcontinuum H of $M/\mathfrak{D}$ such that $d_w \in H^0 \subseteq H \subseteq M/\mathfrak{D} - \{d\}$. But then H^{-1} is a subcontinuum of M containing w in its interior and missing d^{-1}, contradicting the fact that $w \in T(d^{-1})$. Hence $Z(d^{-1}) = d^{-1}$ and the elements of $\mathfrak{D}$ are Z-closed.

Hence M admits a monotone u.s.c. decomposition $\mathfrak{D}_{s-a}$ which is core with respect to the quotient space being semiaposyndetic, and this decomposition is the core decomposition $\mathfrak{D}_Z$. The next theorem enables one to extend the spectrum of decompositions.

THEOREM 3.9. *If A is a subset of the IUC continuum M and $A = Z(A)$ then $A = C(A)$.*

PROOF. Since C is expansive, $A \subseteq C(A)$. So suppose that $C(A) \not\subseteq A$. Let L be a layer of the irreducible subcontinuum I of M such that L^{-1} intersects A but does not lie in A. We consider two cases:

(1) L^{-1} has void interior with respect to I.

(2) L^{-1} has nonvoid interior with respect to I.

Suppose (1). Denote by $[a, b]$ the quotient space of the decomposition of I into layers. Let w be a point of L^{-1} such that $w \in \mathrm{cl}([a, L)^{-1}) \cap \mathrm{cl}((L, b]^{-1})$. Suppose that $w \in L^{-1} - A$. We show that $z \in Z(A)$.

Suppose there exists a subcontinuum H of M such that $w \in H^0 \subseteq H \subseteq M - \{z\}$. Now $H \cap I$ is not connected, for if so, $H \cap I$ is a subcontinuum of I containing w in its interior with respect to I which misses z. But in [10, Theorem 18, p. 26] Thomas showed that I is not aposyndetic at w with respect to z. Hence $H \cup I = M$ and M is not unicoherent. But then $\mathrm{cl}(M - H)$ is connected and $z \in M - H \subseteq \mathrm{cl}(M - H) \subseteq I - \{w\}$. But Thomas also showed that I is not aposyndetic at z with respect to w, and hence a contradiction. Therefore $z \in K(A)$. In like fashion it is shown that $z \in T(A)$ and hence $z \in Z(A)$, a contradiction.

If $w \in L^{-1} - A$ let $z \in A \cap L^{-1}$ and a contradiction is reached by similar arguments.

Suppose (2). Following Vought [**11**, pp. 75–77] there exists an indecomposable subcontinuum N of M which intersects A, $N \not\subseteq A$, and N has interior with respect to I. Let w be a point of $A \cap N$ and $z \in N - (A \cap N)$ where z lies on a different composant of N. Suppose there exists a subcontinuum H of M such that $w \in H^0 \subseteq H \subseteq M - \{z\}$. Hence $H \cap N$ is not connected since N is indecomposable, and therefore $M = H \cup N$ and M is not unicoherent. But then $\mathrm{cl}(M - H)$ is a proper subcontinuum of N with interior, a contradiction. Hence $z \in T(A)$. Likewise $z \in K(A)$ and hence $z \in Z(A) = A$. Therefore $A = C(A)$.

COROLLARY 3.10. $\mathfrak{D}_V \leqslant \mathfrak{D}_C \leqslant \mathfrak{D}_Z$.

COROLLARY 3.11. *If M has property HIUC then $\mathfrak{D}_{hd} \leqslant \mathfrak{D}_{hac} \leqslant \mathfrak{D}_{s-a}$.*

In what follows we generalize results in [**6**] known for hereditarily unicoherent continua.

COROLLARY 3.12. *If the IUC continuum M is semiaposyndetic then M is hereditarily arcwise connected.*

PROOF. Since M is semiaposyndetic $M/\mathfrak{D}_Z = M$. But $\mathfrak{D}_C \leqslant \mathfrak{D}_Z$ and hence $M/\mathfrak{D}_C = M/\mathfrak{D}_Z = M$.

Therefore M is hereditarily arcwise connected.

We now note several results concerning the types of quotient spaces of IUC continua.

THEOREM 3.13. *If $\mathfrak{D}$ is a monotone u.s.c. decomposition of the IUC continuum M such that $M/\mathfrak{D}$ is semiaposyndetic then $M/\mathfrak{D}$ is a*

(1) *point*,

(2) *simple closed curve*, *or*

(3) *semiaposyndetic dendroid.*

PROOF. Suppose (1) and (2) are not true. Semiaposyndesis is preserved by monotone maps, and by Corollary 3.12 $M/\mathfrak{D}$ is hereditarily arcwise connected. Hence we need only show that $M/\mathfrak{D}$ is hereditarily unicoherent.

Suppose not. Hence $M/\mathfrak{D}$ contains a simple closed curve S and $S \neq M/\mathfrak{D}$. Let x and y be points of S. Since $M/\mathfrak{D}$ is semiaposyndetic we assume there is a subcontinuum H of $M/\mathfrak{D}$ such that $x \in H^0 \subseteq H \subseteq M/\mathfrak{D} - \{y\}$. But $H \cup S$ is a nonunicoherent subcontinuum of $M/\mathfrak{D}$ with interior and hence $M/\mathfrak{D} = H \cup S$. But then S has interior, contradicting property IUC of $M/\mathfrak{D}$.

COROLLARY 3.14. *If $\mathfrak{D}$ is a monotone u.s.c. decomposition of the IUC continuum M such that $M/\mathfrak{D}$ is aposyndetic, then $M/\mathfrak{D}$ is a*

(1) *point*,

(2) *simple closed curve*, *or*

(3) *dendrite.*

IV. Decompositions and smoothness. Suppose that M is a nonunicoherent IUC continuum. Then by the last paragraph of [**3**] M is the sum of two irreducible subcontinua A and B such that $A = \mathrm{cl}(M - B)$ and $B = \mathrm{cl}(M - A)$. In [**10**] Thomas has shown that for an irreducible continuum I the decomposition of I into its layers is core with respect to the quotient space being aposyndetic. Hence $\mathfrak{D}_{hac} = \mathfrak{D}_{s-a} = \mathfrak{D}_a$ and hence no further refining of the quotient space is possible. Hence likewise for M, since core decompositions of M are unions of the respective core decompositions of A and B. Therefore we need only consider the unicoherent IUC continua to complete the spectrum of decompositions.

The notion of smoothness was first studied by Charatonik and Eberhart [**2**] for dendroids and was later generalized in [**7**] for continua hereditarily unicoherent at a point. In [**9**] Maćkowiak generalized smoothness to metric continua. His definition follows.

DEFINITION 4.1 (MAĆOWIAK). A continuum N is *smooth at a point p* provided that if $x_1, x_2, x_3, \ldots$ is a sequence of points converging to the point x of N and px is an irreducible subcontinuum of N from p to x, then there exists a sequence $px_1, px_2, \ldots$ of continua, where px_i is irreducible from p to x_i, such that the limiting set of $px_1, px_2, \ldots$ is px.

THEOREM 4.2 (MAĆKOWIAK). *If N is a continuum and p is a point of N then the following are equivalent*:

(1) *N is smooth at p.*

(2) *For each subcontinuum H of N containing p and for each open set V containing H there exists a continuum K such that $H \subseteq K^0 \subseteq K \subseteq V$.*

He then notes that if N is a continuum and p is a point of N then the statement that N is smooth at p implies the following property $(*)$: if y does not cut x from p then N is aposyndetic at x with respect to y. The point *y cuts x from p* provided each subcontinuum of N containing x and p must also contain y.

THEOREM 4.3. *If p is a point of the unicoherent IUC continuum M then property $(*)$ implies that M is smooth at p.*

PROOF. Let K be a subcontinuum of M containing p and let V be an open set containing K. Let z be a point on the boundary of V. For each point w in K, the point z does not cut w from p, and hence there exists a continuum containing w in its interior which misses z. By the compactness of K we have a continuum H_z such that $K \subseteq H_z^0 \subseteq H_z \subseteq M - \{z\}$.

Hence $\{M - H_z \mid z \in \text{bdy } V\}$ is an open cover of bdy V and hence some finite subcollection $M - H_{z_1}, \ldots, M - H_{z_n}$ covers bdy V. But since M is unicoherent and has property IUC the set $H_{z_1} \cap \cdots \cap H_{z_n} = H$ is a subcontinuum of M and $K \subseteq H^0 \subseteq H \subseteq V$.

DEFINITION 4.4. Let p be a point of M and $A \subseteq M$. Then pA denotes the intersection of all subcontinua of M with interior which contain p and A.

LEMMA 4.5. *If M is a unicoherent IUC continuum then pA is a subcontinuum of M.*

PROOF. Now pA is the intersection of a countable collection of continua $H_1, H_2, \ldots,$ each having interior and containing p and A. Let $K_1 = H_1$ and if i is a positive integer greater than 1 let $K_i = H_i \cap K_{i-1}$. Hence each K_i is a subcontinuum of M and $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$. Hence $K = \bigcap_{i=1}^{\infty} K_i$ is a subcontinuum of M containing p and A and therefore $K = pA$.

LEMMA 4.6. *If p is a point of the continuum M then the set function pA is expansive.*

PROOF. Clearly $A \subseteq pA$ and if $A \subseteq B$ each subcontinuum containing p and B contains A.

The following definition is found in **[8]** for hereditarily unicoherent continua. We generalize the definition to unicoherent IUC continua. Note that the definition of pA is different from the usual one, even for hereditarily unicoherent continua, and hence the following definition will not agree with that in **[8]**.

DEFINITION 4.7. Let p be a point of the unicoherent IUC continuum M and $A \subseteq M$. Define $T_p(A) = pA \cap T(A)$. Now T_p is an expansive set function.

LEMMA 4.8. *If A is a subcontinuum of the unicoherent IUC continuum M then so is $T_p(A)$.*

PROOF. Suppose that $pA \cap T(A)$ is not connected. Let x and y be points of $pA \cap T(A)$ which lie on different components of $pA \cap T(A)$ and I be a subcontinuum of $T(A)$ irreducible from x to y. Let $z \in I - pA$ and H be a subcontinuum of M with interior such that $z \notin H$ and $pA \subseteq H$. But $H \cap I$ is connected, contradicting the irreducibility of I.

Let $\mathfrak{D}_{T_p}$ denote the monotone core decomposition of M into T_p-elements. Again note that if $\mathfrak{D}$ is a monotone u.s.c. decomposition of M and $x \in M$ then d_x denotes the element of $\mathfrak{D}$ which contains x.

THEOREM 4.9. *If M is a unicoherent IUC continuum then $M/\mathfrak{D}_{T_p}$ is smooth at d_p.*

PROOF. Suppose x and y are points of M and d_y does not cut d_x from d_p in $M/\mathfrak{D}_{T_p}$. We need only show that $T(d_y^{-1})$ misses d_x^{-1}, for if it does, then there exists a subcontinuum H of M such that $d_x^{-1} \subseteq H^0 \subseteq H \subseteq M - d_y^{-1}$. Then since $\mathfrak{D}_{T_p}$ is upper semicontinuous, $f(H)$ contains d_x in its interior and misses d_y, where f is the induced quotient map. But then $M/\mathfrak{D}_{T_p}$ is aposyndetic at d_x with respect to d_y.

Hence suppose that $T(d_y^{-1})$ intersects d_x^{-1} and let I be a subcontinuum of M irreducible from d_p^{-1} to d_x^{-1} which misses d_y^{-1}. Now $d_p^{-1} \cup I \cup d_x^{-1} \cup T(d_y^{-1})$ contains a subcontinuum J irreducible from p to d_y^{-1}. But J contains a point z of $T(d_y^{-1})$ not in d_y^{-1}, and since $z \notin T_p(d_y^{-1})$ we have $z \notin pd_y^{-1}$. There exists a subcontinuum H of M such that H has interior, $z \notin H$, and $pd_y^{-1} \subseteq H$. By the irreducibility of J, the set $H \cap J$ is not connected, contradicting the fact that M is a unicoherent IUC continuum.

THEOREM 4.10. *If M is a unicoherent IUC continuum, $M/\mathfrak{D}_{T_p}$ is semiaposyndetic.*

PROOF. Let d_x and d_y be elements of $\mathfrak{D}_{T_p}$. If $T(d_y^{-1})$ misses d_x^{-1} then $M/\mathfrak{D}_{T_p}$ is aposyndetic at d_x with respect to d_y. Since by Theorem 4.9 $M/\mathfrak{D}_{T_p}$ is smooth at d_p we may assume that $d_x \neq d_p \neq d_y$.

Suppose that $T(d_y^{-1})$ intersects d_x^{-1}. Let I be an irreducible subcontinuum of $T(d_y^{-1})$ from d_y^{-1} to d_x^{-1}. Let z be a point of $I \cap d_x^{-1}$. Since $z \notin d_y^{-1}$ and $z \in T(d_y^{-1})$ then $z \notin pd_y^{-1}$. Hence there is a subcontinuum H of M with interior such that $pd_y^{-1} \subseteq H$ and $z \notin H$. Hence H misses some point w of $I - [(d_x^{-1} \cup d_y^{-1}) \cap I]$.

We may suppose further that d_x cuts d_y from d_p, for if not, since $M/\mathfrak{D}_{T_p}$ is smooth at d_p the proof is complete. Hence H intersects d_x^{-1} and $(H \cup d_x^{-1}) \cup I$ is a subcontinuum of M with interior and $(H \cup d_x^{-1}) \cap I$ is not connected since $w \notin H \cup d_x^{-1}$. Therefore $T(d_y^{-1})$ misses d_x^{-1}.

THEOREM 4.11. *Suppose $\mathfrak{D}$ is a monotone u.s.c. decomposition of the unicoherent IUC continuum M such that $M/\mathfrak{D}$ is semiaposyndetic and smooth at d_p. Then each element of $\mathfrak{D}$ is T_p-closed.*

PROOF. Let $x \in M$ and, since T_p is expansive, $d_x^{-1} \subseteq T_p(d_x^{-1})$. Suppose there exists a point y in $T_p(d_x^{-1})$ not in d_x^{-1}. Hence $d_x \neq d_y$ but $y \in pd_x^{-1} \cap T(d_x^{-1})$. Since $y \in T(d_x^{-1})$ we may suppose that $d_x \neq d_p$, for if $d_x = d_p$, since $M/\mathfrak{D}$ is smooth at d_p, there exists a subcontinuum H of M such that $d_x^{-1} \subseteq H^0 \subseteq H \subseteq M - d_y^{-1}$. Also y can be chosen such that $y \notin d_p^{-1}$.

Since $y \in T(d_x^{-1})$ we have that d_x cuts d_p from d_y. Since $M/\mathfrak{D}$ is semiaposyndetic there exists a subcontinuum K of M such that $d_x^{-1} \subseteq K^0 \subseteq K \subseteq M - d_y^{-1}$.

Consider $M - K$. Let C be the component of $M - K$ containing p. Hence $\mathrm{cl}(C)$ misses d_y^{-1} since d_x cuts d_p from d_y. But $\mathrm{cl}(C) \cup K$ is a subcontinuum with interior which contains p and d_x^{-1} missing y, a contradiction. Hence $d_x^{-1} = T_p(d_x^{-1})$.

Hence there is a monotone core decomposition of the unicoherent IUC continuum M, $\mathfrak{D}_s$, with respect to the quotient space being semiaposyndetic and smooth at the element containing p, and $\mathfrak{D}_s = \mathfrak{D}_{T_p}$.

COROLLARY 4.12. $\mathfrak{D}_{s-a} \leqslant \mathfrak{D}_s \leqslant \mathfrak{D}_a$.

PROOF. Since $M/\mathfrak{D}_s$ is semiaposyndetic, $\mathfrak{D}_{s-a} \leqslant \mathfrak{D}_s$. If $M/\mathfrak{D}_a$ is aposyndetic then clearly it is semiaposyndetic and property (*) holds immediately.

Therefore we have the spectrum $\mathfrak{D}_{hd} \leqslant \mathfrak{D}_{hac} \leqslant \mathfrak{D}_{s-a} \leqslant \mathfrak{D}_s \leqslant \mathfrak{D}_a$ for unicoherent HIUC continua and the spectrum $\mathfrak{D}_V \leqslant \mathfrak{D}_C \leqslant \mathfrak{D}_Z \leqslant \mathfrak{D}_{T_p} \leqslant \mathfrak{D}_T$ for unicoherent IUC continua, where $\mathfrak{D}_T$ denotes FitzGerald and Swingle's core decomposition into T-closed elements.

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